Phenomenological equation of state and late-time cosmic acceleration

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Assuming a flat Friedmann-Robertson-Walker cosmology with a single perfect fluid, we propose a pressure-density ratio that evolves as a specific universal function of the scale parameter. We show that such a ratio can indeed be consistent with several observational constraints including those pertaining to late-time accelerated expansion. Generic dynamical scalar field models of Dark energy (with quadratic kinetic terms in their Lagrange density) are shown to be in accord with the proposed equation-of-state ratio, provided the current matter density parameter $\Omega_{m0} < 0.23$ - a value *not* in agreement with recent measurements.

The discovery of accelerated expansion of the universe from type Ia Supernova data [1], [2] contingent upon earlier observations [3] indicating that the universe should be spatially flat, has led cosmologists to postulate a new form of energy - the so-called 'Dark' energy - as the dominant source of evolution in the present epoch. To be consistent with the most recent observations, Dark energy is to be thought of as a perfect barytropic cosmic fluid with an equation-of-state (pressure-density) ratio less than -1/3. One successful candidate for this Dark energy is the cosmological constant which can be thought of as a perfect barytropic fluid with a linear equation of state $P_\Lambda = -\rho_\Lambda$ [4].

If one considers the entire evolution of the universe, starting from radiation-matter equilibrium through photon decoupling and matter domination, all the way down to the present era, a single perfect barytropic fluid with a *linear* equation of state $P = \kappa\ \rho$, with a constant κ, is clearly inappropriate as a 'universal' description. The pressure-density ratio κ changes during this evolution from 1/3 at the time of radiation domination, through zero during matter domination to -1/3 or less during the current era, and perhaps to -1 eventually, if the present acceleration is to be eternal. This variation conflicts directly with a constant equation-of-state ratio which implies the density evolution formula

$$\rho(a) \ = \ \rho_0 \ \left(\frac{a_0}{a}\right)^{3[1+\kappa]} \ , \tag{1}$$

where, ρ_0 is the total density in the current epoch; this is so, since, with a cosmological constant, the *lhs* of (1) possesses a constant scale factor independent piece which is clearly absent in the *rhs*. First order perturbations of the type

$$\kappa(\rho) = \kappa + \epsilon\rho \ , \tag{2}$$

where, ϵ is an infinitesimal positive constant, lead to a cosmetic change in (1), given by

$$\rho \ = \ \rho_0 \ \left(\frac{a_0}{a}\right)^{3(1+\kappa)} \exp -\epsilon' \rho_0 \ , \tag{3}$$

where, $\epsilon' \equiv \epsilon/(1+\kappa)$ is also small, and quantities with subscript 0 indicate their values in the current epoch. Clearly, this cannot eliminate the incompatibility discussed above. One can at best expect the equation of state to be *piecewise* linear corresponding to different epochs.

On the other hand, it is tempting to try to model late-time cosmological evolution in terms of a single perfect fluid; such a fluid must have a necessarily *nonlinear* equation of state

$$P \ = \ \kappa(\rho)\ \rho \ , \tag{4}$$

and since the energy density decays with the scale factor, the pressure-density ratio must also evolve with the scale factor, with a functional form chosen to satisfy the 'boundary conditions' discussed above. This 'single fluid' scenario is motivated further by the fact that the evolution of the universe must necessarily be incremental in nature, especially in the later epochs. It is difficult to conceive of a *sudden* impulsive change in the equation of state at a specific value of the redshift. Rather what is far more likely is a gradual transition, as the universe expands, between 'phases' where the dominant driving source is one or other kind of energy (including Dark energy) for a different ranges of values of the redshift. Thus, from a classical phenomenological standpoint, it is worthwhile to try and model cosmic evolution in late times by a single perfect fluid with an equation-of-state ratio that changes continuously as a function of the scale factor. With such an evolving κ, it is not easy to characterise what one means by 'radiation domination' or 'matter domination' in the appropriate epoch. On the positive side, Dark energy is placed on the same footing as

radiation or dust matter in this scenario. This demystification may be desirable for observational/phenomenological purposes.

In this paper, we choose a specific one-parameter family of functions of the scale factor a for the pressure-density ratio; we use several pieces of cosmological data to constrain this free parameter, and a viable phenomenological window seems to emerge. We should mention that the description that we have so far is a coarse-grained one wherein details of cosmic events prior to photon decoupling have been washed away. We find it significant though that such a simple scenario is consistent with recent cosmological findings. We should point out that model-independent equations of state for dark energy have indeed been discussed recently in the literature [5]. Our proposal for the equation of state involves not just the era of cosmic acceleration, but the entire evolution at late-times, starting roughly with radiation-matter decoupling.

Our choice for the functional form of the pressure-density ratio is

$$\kappa(a) \;=\; \tanh\left(\frac{a_m}{a_0} \;-\; \frac{a}{a_0}\right) \;. \tag{5}$$

where a_0 is the present value of the scale factor and a_m is the only free parameter in this equation of state. This can be reexpressed in terms of the redshift (with $a/a_0 \equiv (1+z)^{-1}$),

$$\kappa(z) \;=\; \tanh\left(\frac{1}{1+z_m} \;-\; \frac{1}{1+z}\right) \;, \tag{6}$$

where z_m is the redshift distant corresponding to the value of the scale factor a_m. In fig. 1 below, we plot this nonlinear equation-of-state ratio $\kappa(z)$ as a function of the redshift distance z for a band of values of the parameter z_m.

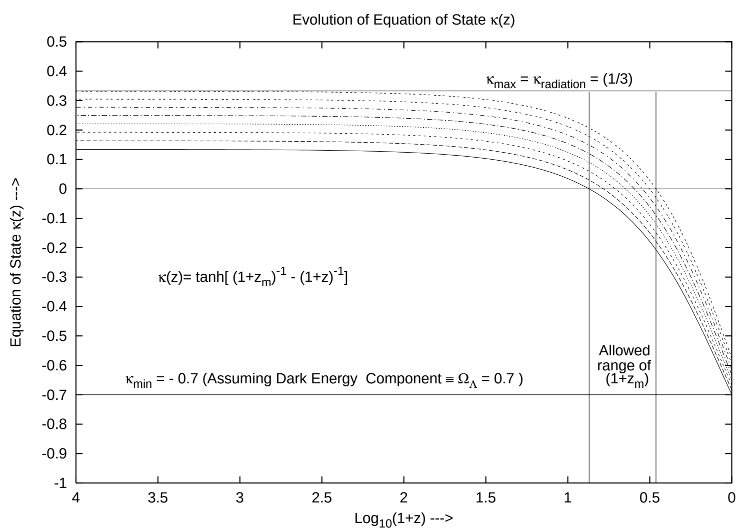

Figure 1. Evolution of $\kappa(z)$ vs. z for $(a_m/a_0) = 0.135$ *to* 0.34.

The family of curves of the equation-of-state ratio κ as a function of the redshift z shown in Fig. 1 corresponds to the 'allowed region': the uppermost and lowermost curves of $\kappa(z)$ correspond respectively, as shown on the graph, to κ_{max} depicting (partial) radiation domination at $z = 10^4$ or higher, and to κ_{min} corresponding to the domination of Dark energy (about 70% of the total energy density for a flat FRW universe, according to recent observations [4]) in the present epoch (indicated by the subscript 0). The 'allowed region' in Fig. 1 restricts the parameter a_m/a_0 (or alternatively, z_m) in our expression (5) for the equation-of-state ratio to the range $[0.175, 0.34]$. It remains to be seen whether observed values of other cosmological parameters constrain this range even further.

For brevity of expression, we rescale a by a_0 in the sequel. In terms of the rescaled a, the overall density $\rho(a)$ evolves, via the Friedmann equations appropriate to a flat FRW universe, according to

$$\frac{\rho(a)}{\rho_0} \;=\; \left[a^{-1}\exp\int_a^1 \frac{d\hat{a}}{\hat{a}}\;\kappa(\hat{a})\right]^3 \;, \tag{7}$$

where $\kappa(a)$ is given, in its turn, by (5). The first factor in (7) exhibits the evolution of the standard density parameter for pressureless dust matter in absence of any Dark energy. Thus, our proposal (5) of the scale factor dependent

equation-of-state ratio $\kappa(a)$ leads to the additional correction factor which incorporates the effect of Dark energy. We have of course assumed here that the total density in the current epoch is precisely equal to the critical density. Eq. (7) will be useful in examining possible constraints on the free parameter a_m from cosmological data.

One further constraint is the product $H_0 t_0$ of the current value of the Hubble parameter and the age of the universe [6]. The Friedmann equations with a flat FRW ansatz leads to the relation

$$H_0 \; t_0 \;=\; \int_0^1 \frac{da}{a} \left(\frac{\rho(a)}{\rho_0}\right)^{-1/2} , \tag{8}$$

where, $\rho(a)$ is given by (7) and, as earlier, $\kappa(a)$ is given by (5). The *lhs* of (8) is known to lie in the range 0.93 ± 0.20. according to measurements reported in [6]. The nested integrals on the *rhs* of (8) can be performed numerically to extract an 'allowed range' of values for the free parameter a_m/a_0. Unfortunately, it turns out that this range is actually *larger* than the allowed range extracted above from partial radiation domination at redshifts larger than 10,000 and from the current observed value of 70% of the Dark energy density parameter for a flat universe. This range of allowed values is plotted in Fig. 2 below, alongside constraints from other sources, as we now proceed to explain.

Perhaps the strongest constraint on a_m comes from the angular position of the first acoustic ('Doppler') peak in the angular distribution of the Cosmic Microwave Background Radiation (CMBR) temperature. Anisotropic temperature fluctuations arise due to fluctuations in the gravitational potential due to density fluctuations in the cosmic fluid. The latter, in their turn, produce pressure fluctuations that propagate as sound waves. As the universe expands, pressure fluctuations decrease due to changes in the nature of the cosmic fluid, leading to attenuation of sound waves. The proper distance travelled by such waves signifies a sonic horizon, whose existence is inferred through the first (and largest) peak in the angular distribution spectrum of CMBR. Such a peak has already been observed in the anisotropic temperature fluctuations of the CMBR by several groups starting with COBE [7] through BOOMERANG and MAXIMA [3], [8], [9]. Our task here is to compute the angular position of the first acoustic peak within the single fluid picture with our proposed equation-of-state ratio (5), as a function of the free parameter a_m. Comparison with recent data on this acoustic peak is then used to constrain a_m. The reason that this constraint is the strongest is linked to the fact that the CMBR temperature anisotropy has been (and is being) probed to an accuracy of better than 10^{-5}.

The location of the first acoustic peak is expressed in terms of the angle θ subtended by the *sonic horizon*. This is defined as [10], [11]

$$\theta \;\equiv\; \frac{d_{sh}}{d_A} , \tag{9}$$

where, d_{sh} is the radius of the sonic horizon, defined as the proper distance traversed by sound waves in the fluid medium until they are attenuated due to reduction in pressure fluctuations, and d_A is the proper angular diameter distance of the sonic horizon. d_{sh} is given by $d_{sh} = a_s \, r_{sh}$ where a_s is the scale factor at which sound stops propagating: $c_s(a_s) = 0$. This can be computed from the coordinate size r_{sh} given by

$$r_{sh} \;=\; \int_{a_s}^1 da \, \frac{c_s}{\dot{a}} , \tag{10}$$

where, the velocity of sound c_s, defined in the standard fashion as $c_s^2 \equiv dP/d\rho$, is given, in our model by,

$$c_s^2(a) \;=\; \frac{a}{3} \;+\; \kappa(a) \; (1 \;-\; \frac{a}{3}) . \tag{11}$$

It is easy to see that $c_s(a_s) = 0$ gives an analytical relation between a_m and a_s,

$$a_m \;=\; a_s \;-\; \tanh^{-1}\left[\frac{a_s/3}{1 - a_s/3}\right] . \tag{12}$$

The angular diameter distance is given by the length of the null geodesic traversed by photons travelling from the sonic horizon towards the earth, scaled by the scale factor at sound attenuation: $d_A = a_s \; r_p$, with the coordinate path length r_p being given by

$$r_p \;=\; \int_{a_s}^1 \frac{da}{a \, \dot{a}} . \tag{13}$$

Substituting eq.s (10) and (13) into (9), we obtain

$$\theta(a_s) \;=\; \frac{\int_0^{a_s} \; da \; \frac{c_s}{\dot a}}{\int_{a_s}^{1} \; da \frac{1}{a\dot a}} \; . \tag{14}$$

The order l_1 of the multipole contributing to the first acoustic peak is related to θ by the relation $\theta(a_s) = 2\pi/l_1$. Recent measurements of functions of the temperature distribution have constrained l_1 to be in the range 212 ± 17. Using the formula (14) for $\theta(a_s)$, one can determine (numerically) the range of allowed values of a_s and hence, using (12), obtain the admissible range of values of a_m. One obtains the range $a_m = 0.172 \pm 0.005$, which falls well within the range allowed by the other cosmological constraints. In fig. 2 we exhibit these allowed ranges of values for this parameter constrained by cosmological data.

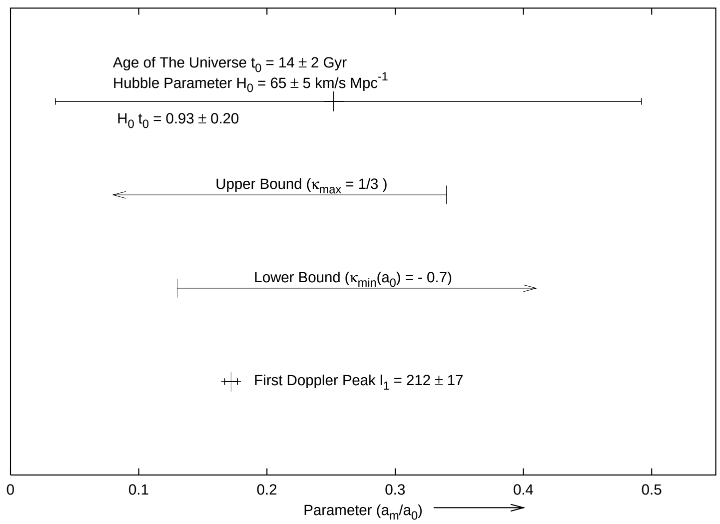

Figure 2. Allowed range of a_m constrained by cosmological data.

Having established the viability of the phenomenological equation-of-state ratio given by (5), the issue now arises whether such an equation of state can actually be obtained from a dynamical scalar field model. In this letter we confine our attention to models with a standard quadratic kinetic energy term; tachyonic models now emerging as popular candidates for Dark energy have non-standard actions with infinite-order derivative terms and lie outside the purview of this letter. The same is the case with 'tracker' models with exotic kinetic energy structures. We shall consider these in a future assay.

We assume that at late times, the major contribution to the total energy density comes from pressureless dust matter and the scalar field such that $\rho \;=\; \rho_m + \rho_\phi$. Assuming a spatially homogeneous scalar field with a potential $V(\phi)$, the density and pressure are given by

$$\begin{aligned} \rho_\phi \;&=\; \frac{1}{2}\dot\phi^2 \;+\; V(\phi) \\ P_\phi \;&=\; \frac{1}{2}\dot\phi^2 \;-\; V(\phi) \; . \end{aligned} \tag{15}$$

This implies that

$$\frac{1}{2}\dot\phi^2 \;=\; V \;+\; \kappa(a)\rho(a) \; . \tag{16}$$

The scalar field obeys the field equation (in a flat FRW geometry)

$$\ddot\phi \;+\; 3(\frac{\dot a}{a}) \; \dot\phi \;+\; \frac{dV}{d\phi} \;=\; 0 \; . \tag{17}$$

The potential $V(\phi)$ can be eliminated from the field equation using eq.s (15) and (16), yielding a first order linear differential equation in $\dot\phi^2$

$$\frac{d}{dt}[a^3\ \dot{\phi}^2]\ =\ a^3\ \frac{d}{dt}[\kappa\rho]\ . \tag{18}$$

The solution may be written as

$$\frac{1}{2}\dot{\phi}^2\ =\ \frac{1}{2}\left[\frac{\rho_{D0}}{a^3} + \kappa(a)\rho(a) + \frac{3}{a^3}\int_a^1 dx x^2 \kappa(x)\rho(x)\right]\ , \tag{19}$$

where, $\rho_{D0} \equiv \rho_0 - \rho_{m0}$ is the density of Dark energy at the present epoch.

Now, in the range $[a_m\ ,\ 1]$ of the scale factor, $\kappa(a)$ is clearly negative, so that the second and third terms on the *rhs* of eq. (19) produce negative contributions, while the first term contributes positively. Since the *lhs* must always be positive, this leads (numerically) to a rather stringent constraint on the Dark energy density parameter at the present epoch: $\Omega_{D0}\ >\ 0.77$. That is to say, *a generic scalar field model of Dark energy can be consistent with the equation of state (5) only for a rather large Dark energy density parameter, or what is equivalent, a current matter density parameter* $\Omega_{m0} < 0.23$. This is lower than the measured value of around 1/3.

It follows that, if our phenomenological equation of state is correct, a generic dynamical scalar field model of Dark energy (like, for instance, some Quintessence models), with quadratic kinetic energy, is unlikely to be an appropriate description. In other words, scalar field models may not be able to explain why there exists a cosmological constant that is so small when compared to the vacuum energy of the standard model, and is yet almost the sole driving force of cosmic evolution at present. That the proposed equation-of-state ratio is consistent with recently measured values of several key cosmological parameters lends more credence to this speculation.

We thank G. Date for several useful discussions. We also thank A. Ganguly for discussions and for bringing ref. 5 to our attention.

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